Exam duration — Three hours
Reading time — 15 minutes

This paper has 6 pages, including this cover sheet.

Instructions to Invigilators:
Initially, students are to receive a 14 page script book.
Students may take the exam paper with them at the end of the examination.

Authorized Materials:
No calculators, computers or mobile phones are permitted.
No written or printed materials may be brought into the examination room.

Instructions to Students:
There is a formula sheet on page 6 of this examination paper.
There are 10 questions on this examination paper.
All questions may be answered.
Marks for each question are indicated on the paper.
The total number of marks on the exam paper is 145.

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1. (a) Calculate the following limits, if they exist. Explain your answers using the limit laws, continuity, sandwich theorem, or other appropriate arguments. If the limit does not exist, explain why it does not exist.

(i) \[ \lim_{x \to 2} \frac{x^2 + x - 6}{x^2 - 3x + 2} \]

(ii) \[ \lim_{t \to \infty} \frac{2}{\sqrt{t^2 + t - t}} \]

(iii) \[ \lim_{x \to 0^+} \frac{\log(x + 1)}{\sqrt{x}} \]

(b) Consider the function \( f(x) \) given by

\[
f(x) = \begin{cases} 
2e^x, & x < -1 \\
x^2 + x + 1, & -1 \leq x < 1 \\
c \sin\left(\frac{\pi}{6}x\right) - 1, & x \geq 1
\end{cases}
\]

(i) Explain briefly why \( f(x) \) is continuous whenever \( x \neq -1 \) and \( x \neq 1 \).

(ii) Is \( f(x) \) continuous at \( x = -1 \)? Justify your answer.

(iii) For what value of the constant \( c \) is \( f(x) \) continuous at \( x = 1 \)? Justify your answer.

2. (a) Find all values of \( x \) satisfying \( \cosh x = 3 \). Write your answers in terms of logarithms.

(b) (i) Show that, for \(-1 < x < 1\),

\[
\frac{d}{dx} (\text{arctanh } x) = \frac{1}{1 - x^2}.
\]

(ii) Explain why \( \frac{d}{dx} (\text{arctanh } x) \) is only defined for \(-1 < x < 1\).

(iii) If \( g(x) = \text{arctanh } (\log x) \),

find \( g'(x) \).

(iv) For what values of \( x \) is \( g'(x) \), found in part (iii), defined? Justify your answer.
3. (a) Use the complex exponential to prove the identity
\[
\cos 3\theta = 4 \cos^3 \theta - 3 \cos \theta.
\]
(b) Find all complex solutions to the equation
\[
z^4 = \frac{2}{1 + i}.
\]
Write your answers in the form \(re^{i\theta}\), where \(-\pi < \theta \leq \pi\).
(c) Use the complex exponential to evaluate the derivative
\[
\frac{d^6}{dt^6} \left(e^{2t} \sin 2t\right).
\]

[15 marks]

4. Evaluate the following integrals.
(a) \[\int \frac{1}{(x^2 + 16)^{3/2}} \, dx\]
(b) \[\int \arcsin x \, dx\]

[14 marks]

5. Consider the differential equation
\[
2xy \frac{dy}{dx} = x^2 + 3y^2. \tag{1}
\]
(a) Show that the substitution \(u = \frac{y}{x}\) reduces the differential equation to
\[
2ux \frac{du}{dx} = 1 + u^2.
\]
(b) Hence find the solution \(y(x)\) of the original differential equation (1) with initial condition \(y(1) = 2\).

[11 marks]
6. A 700-litre tank initially contains 50 litres of brine solution, consisting of 12 kg of salt dissolved in water. Beginning at time zero, brine containing 2 kg of salt per litre is added at the rate of 3 litres per minute. The solution is mixed evenly and is poured out of the tank at the rate of 2 litres per minute.

(a) Show that the differential equation governing the quantity of salt, \( x(t) \) kg, in the tank after time \( t \) minutes is given by

\[
\frac{dx}{dt} = 6 - \frac{2x}{50 + t}.
\]

What is the initial condition for \( x(t) \)?

(b) Solve the initial value problem in part (a) to find \( x(t) \).

(c) How much salt is in the tank when it contains 100 litres of brine?

[14 marks]

7. Consider the differential equation

\[
\frac{d^2y}{dx^2} - 4 \frac{dy}{dx} + 4y = f(x).
\]

(a) Find the general solution if \( f(x) = 0 \).

(b) Find a particular solution if \( f(x) = x - 1 \).

(c) Find a particular solution if \( f(x) = e^{2x} \).

(d) Hence find the general solution if \( f(x) = 3x - 3 - 5e^{2x} \).

[16 marks]

8. An electric circuit consists of a resistor with a resistance of 12 ohms, a capacitor with a capacitance of 0.1 farads and an inductor with an inductance of 2 henry connected in series with a voltage source of \( 52 \cos t \) volts. Initially the charge on the capacitor is 3 coulombs and the current in the circuit is zero.

(a) Using Kirchhoff’s voltage law, show that the charge \( q(t) \) coulombs on the capacitor at time \( t \) seconds satisfies

\[
\frac{d^2q}{dt^2} + 6 \frac{dq}{dt} + 5q = 26 \cos t.
\]

(b) Write down \( q(0) \) and \( \dot{q}(0) \).

(c) Determine the charge on the capacitor at any time.

(d) Describe the long term behaviour of the charge on the capacitor. Identify the steady-state solution and transient solution.

[20 marks]
9. Consider the function

\[ f(x, y) = e^{x^2 - y}. \]

(a) Verify that

\[ \frac{\partial f}{\partial x} + 2x \frac{\partial f}{\partial y} = 0. \]

(b) Find the equation of the tangent plane to the surface \( z = f(x, y) \) at the point \((-1, 1, 1)\).

(c) Write down the gradient vector of \( f \) at the point \((-1, 1)\).

(d) Show that the rate of change of \( f \) at the point \((-1, 1)\) in the direction of the vector \((2, -4)\) is equal to zero.

(e) Use the result of part (d) above to find a vector in the \( xy \)-plane which is tangent to the level curve \( f(x, y) = 1 \) at the point \((-1, 1)\). Give a brief explanation.

[10 marks]

10. (a) Consider the function

\[ f(x, y) = 2x^2y + xy + \frac{2}{3}y^3. \]

(i) Calculate the first and second order partial derivatives \( f_x, f_y, f_{xx}, f_{xy}, f_{yx}, f_{yy} \).

(ii) Find and classify all the stationary points of \( f(x, y) \).

(iii) Evaluate the double integral

\[ \int_{-1}^1 \int_0^1 f(x, y) \, dx \, dy. \]

(b) Let \( h(x, y) \) be a differentiable function satisfying

\[ h(tx, ty) = t^k h(x, y) \quad \text{for all real } x, y, t, \tag{2} \]

where \( k \) is a positive integer. Prove the identity

\[ xh_x(x, y) + yh_y(x, y) = kh(x, y) \quad \text{for all real } x, y. \]

[Hint: First differentiate equation (2) with respect to \( t \).]

[20 marks]

END OF EXAMINATION - FORMULA SHEET ON NEXT PAGE

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Standard Integrals
\[ \int \sin x \, dx = -\cos x + C \]
\[ \int \cos x \, dx = \sin x + C \]
\[ \int \sec^2 x \, dx = \tan x + C \]
\[ \int \csc^2 x \, dx = -\cot x + C \]
\[ \int \sinh x \, dx = \cosh x + C \]
\[ \int \cosh x \, dx = \sinh x + C \]
\[ \int \text{sech}^2 x \, dx = \tanh x + C \]
\[ \int \text{cosech}^2 x \, dx = -\coth x + C \]
\[ \int \frac{1}{\sqrt{a^2 - x^2}} \, dx = \arcsin \left( \frac{x}{a} \right) + C \]
\[ \int \frac{1}{a^2 + x^2} \, dx = \frac{1}{a} \arctan \left( \frac{x}{a} \right) + C \]
\[ \int \frac{1}{\sqrt{x^2 - a^2}} \, dx = \text{arccosh} \left( \frac{x}{a} \right) + C \]
\[ \int \frac{1}{\sqrt{x^2 + a^2}} \, dx = \text{arcsinh} \left( \frac{x}{a} \right) + C \]

where \( a > 0 \) is constant and \( C \) is an arbitrary constant of integration.

Useful Formulae
\[ \cos^2 x + \sin^2 x = 1 \]
\[ 1 + \tan^2 x = \sec^2 x \]
\[ \cot^2 x + 1 = \csc^2 x \]
\[ \cos 2x = \cos^2 x - \sin^2 x \]
\[ \cos 2x = 2 \cos^2 x - 1 \]
\[ \cos 2x = 1 - 2 \sin^2 x \]
\[ \sin 2x = 2 \sin x \cos x \]
\[ \cosh x = \frac{1}{2} (e^x + e^{-x}) \]
\[ e^{ix} = \cos x + i \sin x \]
\[ \cos x = \frac{1}{2} (e^{ix} + e^{-ix}) \]
\[ \sin x = \frac{1}{2i} (e^{ix} - e^{-ix}) \]

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