Graph Theory 620-352

Course Outline

Broadly speaking, there are three types of information given in lectures:


The number of lectures allocated to each topic is an approximation. The actual time used may be shorter or longer than indicated.

1. Introduction [Week 1, 3 Lectures]

Definitions graph, vertex, vertex set, edge, edge set, adjacent vertices, edge joins vertices, multigraph, pseudograph, loop, order and size of a graph, degree of a vertex, neighbourhood of a vertex, edges incident to a vertex, isolated-vertex, end-vertex, r-regular graph, regular graph, degree sequence, minimum degree, maximum degree, isomorphic graphs, isomorphism.

Theorems, Lemmas and Corollaries maximum number of edges in a graph, Handshaking Theorem, sum of vertex degrees is even, no odd number of odd degrees, two vertices of same degree, degrees of vertices of isomorphic graphs, average degree is between $\delta(G)$ and $\Delta(G)$, properties of a graph preserved by isomorphism.

1. Introduction (cont’d) [Week 2, 3 Lectures]

Definitions subgraph, induced subgraph, deletion of a vertex, spanning subgraph, complement of a graph, self-complementary graph, walk, trail, path, length of walk, cycle, closed walk, connected graphs, disconnected graphs, components, number of components, cut-vertex, bridge, complete graph, Petersen graph, hypercube, paths, cycles, bipartite graph, complete bipartite graph, union, join, Cartesian product.

Theorems, Lemmas and Corollaries every walk contains a path, cycles contain two paths between pairs of vertices, bridges are not in cycles, bipartite graphs and no odd cycles.

Algorithms finding a path in a walk.

2. Algorithms [Week 3, 3 Lectures]

Definitions algorithms, (worst case) complexity of an algorithm, order of a function, polynomial algorithms, searching problem, adjacency matrix, sorting problem.

Theorems, Lemmas and Corollaries complexity of binary search.

Algorithms sequential search algorithm, binary search algorithm, bubblesort algorithm, adjacency listing.

3. Trees and distance [Week 4, 3 Lectures]
Definitions tree, forest, binary search tree, complete m-ary tree, complete binary tree, spanning trees, fundamental cycle, directed tree, rooted tree, child, descendant, parent, ancestor, depth-first search tree, dfs(v)

Theorems, Lemmas and Corollaries spanning trees exist if and only if graph is connected, characterizations of a tree (equivalent definitions), unique path between two vertices in a tree, each edge is a bridge in a tree, every connected graph contains a spanning tree, exact one cycle when adding an edge to a tree

Algorithms depth-first search algorithm

3. Trees and distance (cont’d) [Week 5, 2 Lectures, Good Friday]

Definitions minimum spanning tree, breadth-first search forest, weighted graph

Theorems, Lemmas and Corollaries DFS characterisation of cut-vertices, DFS characterisation of bridges, Kruskal finds minimum spanning tree

Algorithms finding cut-vertices using DFS, finding bridges using DFS, breadth-first search algorithm Kruskal’s algorithm

3. Trees and distance (cont’d) [Week 6, 3 Lectures]

Definitions distance, eccentricity of a vertex, diameter, radius, centre of a graph

Theorems, Lemmas and Corollaries Prim finds minimum spanning tree, distance function is a metric, \( rad(G) \leq diam(G) \leq 2 rad(G) \), centre of a tree is \( K_1 \) or \( K_2 \)

Algorithms Prim’s algorithm

3. Trees and distance (cont’d) [Week 7, 1 Lecture]

Definitions shortest path problem

Algorithms Dijkstra’s algorithm

4. Matchings and factors [Week 7, 2 Lectures]

Definitions marriage problem, assignment problem, matching, perfect matching, matched vertex, maximum matching, alternating path, augmenting path factor 1-factor 1-factorization 1-factorization

Theorems, Lemmas and Corollaries Berge’s Theorem (maximum matching if and only if no augmenting path), Tutte’s Theorem [No proof], every bridgeless cubic graph contains a perfect matching

4. Matchings and factors (cont’d) [Week 8, 2 Lectures, ANZAC Day]

Definitions augmentation, alternating tree, maximum alternating tree

Theorems, Lemmas and Corollaries Hall’s marriage theorem

Algorithms maximum matching algorithm for bipartite graphs

5. Eulerian graphs [Week 9, 3 Lectures]
**Definitions** Eulerian trails, Eulerian circuits, Eulerian multigraphs, Chinese postman problem

**Theorems, Lemmas and Corollaries** Eulerian multigraphs and even degree vertices, Eulerian trails and at most two odd degree vertices, Chinese postman solution theorem

**Algorithms** Finding eulerian circuits

6. Network flows and connectivities [Week 10, 3 Lectures]

**Definitions** digraph (directed graph), arc, network, capacity function, capacity, flow, val \( f \): value of \( f \), maximum flow, cut, capacity of cut, minimum cut, \( f \)-saturated arc, \( f \)-zero arc, \( f \)-augmenting semipath

**Theorems, Lemmas and Corollaries** flow conservation theorem, comparison of val \( f \) to capacity of cut, max-flow if and only if no augmenting semipath, Max-flow Min-cut Theorem

**Algorithms** Ford-Fulkerson algorithm

6. Network flows and connectivities [Week 11, 3 Lectures]

**Definitions** \( \kappa(G) \): connectivity, internal disjoint path, edge-disjoint path \( \lambda(G) \): edge-connectivity

**Theorems, Lemmas and Corollaries** \( \kappa \leq \lambda \leq \delta \), Menger’s Theorem

**Algorithms** labelling algorithm for network flow problem

7. Colouring and planarity [Week 12, 3 Lectures]

**Definitions** vertex colouring, chromatic number, edge-colouring, edge chromatic number, planar graphs, subdivision

**Theorems, Lemmas and Corollaries** Brooks’ Theorem, Vizing’s Theorem, Euler’s formula, every planar graph of order \( n \geq 3 \) has at most \( 3n - 6 \) edges, every planar graph contain a vertex of degree at most 5, \( K_5 \) and \( K_{3,3} \) are nonplanar, Kuratowski’s Theorem, Four Colour Theorem